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FLOW OF A VISCOUS LIQUID FILM ON THE SURFACE OF A ROTATING DISK

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Results are presented from numerical calculations of steady-state axisymmetric flow of a film of viscous incompressible liquid over the surface of a plane rotating disk.

Film flow of a liquid over the surface of a rotating disk is found in many technological processes, the calculation of which requires knowledge of the hydrodynamic characteristics of such a flow. A number of theoretical and experimental studies have been dedicated to this question [1-5]. Dorfman [1] presented results of calculations by the difference method for the case of uniform initial velocity component profiles, [2-4] considered asymptotic solutions for relatively thin films, while [5] numerically determined a solution of special form. The present study will use the colocation method of [3], which allows calculations for a wide range of parameter values.

Let a viscous incompressible liquid be supplied near the axis of rotation of the disk at a constant volume flow rate Q. In analogy to [3], the velocity components u_r, u_θ, u_z in a fixed cylindrical coordinate system r, θ, z fixed to the center of rotation of the disk are represented in the form

$$u_r = \omega r \delta^2 u,\ u_\theta = \omega r (1 + \delta^2 v),\ u_z = \omega H_0 \delta^2 w.$$

The quantity δ appearing in $\sqrt{\nu/\omega}$ is the thickness of the boundary layer which develops near an infinitely large disk rotating in an infinite liquid volume [6].

Without considering surface tension the system of equations and boundary conditions describing steady-state axisymmetric flow of the film, to the accuracy of terms of the order $(H_0/r)^2$, has the form [3]:

$$\frac{\partial u}{\partial x} + 2u + \frac{\partial w}{\partial y} = 0, \qquad (1)$$

$$\frac{\partial^2 u}{\partial y^2} + 1 + 2\delta^2 v - \delta^4 \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial y} + u^2 - v^2 \right) = 0, \qquad (2)$$

$$\frac{\partial^2 v}{\partial y^2} - 2\delta^2 u - \delta^4 \left(u \frac{\partial v}{\partial x} + w \frac{\partial v}{\partial y} + 2uv \right) = 0, \qquad (3)$$

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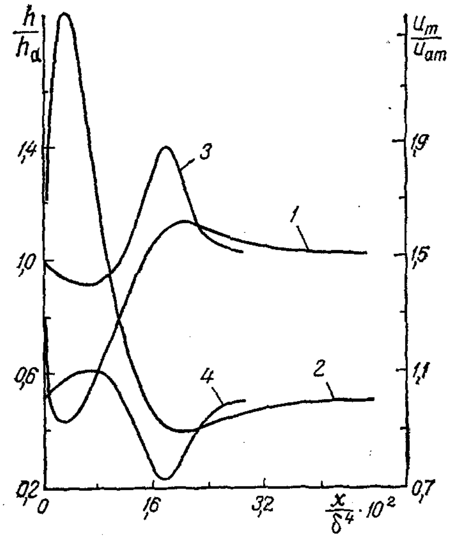

Fig. 1

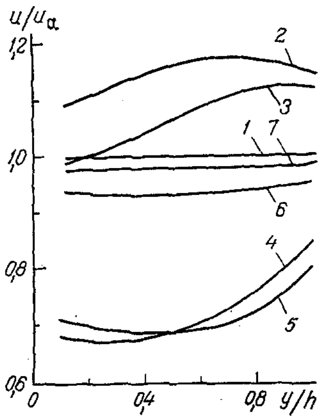

Fig. 2

Fig. 1. Thickness and mean radial velocity vs radius: 1, 2) $\delta = 2.504$; $U = 0.1 \sin(\pi y/2)$, $V = \sin(\pi y/2)$; 3, 4) $\delta = 3.107$, $U = y - y^2/2$, $V = 0$; 1, 3) thickness; 2, 4) mean radial velocity; number of flow lines $N = 10$.

Fig. 2. Development of radial velocity profile at $\delta = 3.107$, $U = y - y^2/2$, $V = 0$: 1) $x = 0$: 2) $0.005\delta^4$; 3) $0.01\delta^4$; 4) $0.015\delta^4$; 5) $0.02\delta^4$; 6) $0.025\delta^4$; 7) $0.029\delta^4$; number of flow lines $N = 10$.

$$y = h(x):\ u\frac{\partial h}{\partial x} = w,\quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$y = 0:\ u = v = w = 0, \tag{5}$$

where h is the film thickness; $x = \ln(r/R)$, $y = z/H_0$. Here Eq. (1) is the continuity equation; Eqs. (2), (3) are the equations of motion for the radial and azimuthal velocity components, respectively; Eq. (4) expresses the kinematic condition and the equality to zero of tangent stresses in two directions on the free surface; Eq, (5) expresses the adhesion and impermeability conditions on the disk surface.

The spreading of the film is considered as a Cauchy problem with initial conditions as formulated below for $x = 0$.

For a numerical solution we introduce flow lines $y = h_n(x)$ and values of the velocity components thereon $u_n(x) = u(x, h_n(x))$, $v_n(x) = v(x, h_n(x))$, $n = 1, 2, \ldots, N$, while $h_N \equiv h$. For the volume flow rates $q_n(x)$, defined by the expressions

$$q_n(x) = \int_{h_{n-1}}^{h_n} u\,dy,\quad n = 1, 2, \ldots, N,\ h_0 \equiv 0, \tag{6}$$

from continuity equation (1) and conditions of nonflow across flow lines, analogous to the kinetic condition of Eq. (4), it follows that

$$\frac{dq_n}{dx} + 2q_n = 0,\ n = 1, 2, \ldots, N. \tag{7}$$

Using the trapezoid expression of [7] to calculate the integrals of Eq. (6), from Eq. (7) we obtain

$$\frac{dh_n}{dx} = \frac{dh_{n-1}}{dx} - (h_n - h_{n-1})\left[\frac{1}{u_n + u_{n-1}}\left(\frac{du_n}{dx} + \frac{du_{n-1}}{dx}\right) + 2\right], \tag{8}$$

$$n = 1, 2, \ldots, N,\ u_0 \equiv 0.$$

The equations of motion written for the flow lines give

$$\frac{du_n}{dx}=\frac{1}{u_n}\left[v_n\left(v_n+\frac{2}{\delta^2}\right)+\frac{1}{\delta^4}\left(\frac{\partial^2 u}{\partial y^2}\bigg|_{y=h_n}+1\right)\right]-u_n,$$
$$\frac{dv_n}{dx}=\frac{1}{\delta^4 u_n}\frac{\partial^2 v}{\partial y^2}\bigg|_{y=h_n}-2\left(v_n+\frac{1}{\delta^2}\right),\ n=1,\ 2,\ \ldots,\ N. \tag{9}$$

To calculate the second derivatives appearing on the right side of Eq. (9) we use the tau-approximation of [8] employing mixed Chebyshev polynomials of the first sort $\varphi_k(\eta)$ defined by the expressions [9]

$$\varphi_1=1,\ \varphi_2=2\eta-1,\ \varphi_k=2\varphi_2\varphi_{k-1}-\varphi_{k-2},\ k=3,\ 4,\ \ldots,\ N+2.$$

Then for the velocity components, for example u, we construct the approximating function

$$u_N(y)=\sum_{k=1}^{N+2}a_k\varphi_k(y/h),$$

the expansion coefficients of which, a_k, k = 1, 2, ..., N + 2, are solutions of a system of linear algebraic equations

$$\sum_{k=1}^{N+2}a_k\varphi_k(h_n/h)=u_n,\ n=1,\ 2,\ \ldots,\ N, \tag{10}$$

$$\sum_{k=1}^{N+2}a_k\varphi_k(0)=0,\ \sum_{k=1}^{N+2}a_k\varphi_k'(1)=0, \tag{11}$$

where Eq. (10) is the condition of equality of the function u_N to the values of the velocity components u on the flow lines, while Eq. (11) is an approximation of boundary conditions on the disk and the film surface. The expressions for the second derivatives of the function u on the flow lines have the form

$$\frac{\partial^2 u}{\partial y^2}\bigg|_{y=h_n}=\frac{d^2u_N}{dy^2}\bigg|_{y=h_n}=\frac{4}{h^2}\sum_{k=1}^{N}b_k\varphi_k(h_n/h),\ n=1,\ 2,\ \ldots,\ N,$$

where

$$b_1=\frac{1}{2}\sum_{\substack{j=3\\ j-\text{ odd}}}^{N+2}(j-1)^3a_j,\ b_k=\sum_{\substack{j=k+2\\ j+k-\text{even}}}^{N+2}(j-1)[(j-1)^2-$$
$$-(k-1)^2]a_j,\ k=2,\ 3,\ \ldots,\ N.$$

To Eqs. (8), (9) we add the initial conditions

$$h_n(0)=n/N,\ u_n(0)=U(h_n),\ v_n(0)=V(h_n),\ n=1,\ 2,\ \ldots,\ N,$$

where U(y), V(y) are specified functions. The value of the film thickness at x = 0 is considered as a characteristic scale.

Integration of Eqs. (8), (9) was performed by a second-order accuracy Adams–Beshfort method [8]. The quantization error along x can be neglected because of the small size of the integration step, specified by the condition for stability of the method. The accuracy of the calculation is determined by the number of flow lines N and the complexity of the velocity component profiles. Equality to zero of the coefficients a_k for higher polynomials was monitored during the calculations.

It follows from Eqs. (1), (4), and the axisymmetric nature of the flow that

$$q(x)\equiv\int_0^h u\,dy=\frac{Q\nu}{2\pi r^2\omega^2H_0^3}.$$

At low values of the parameter δ the problem of Eqs. (1)-(5) without initial conditions has a solution, the main terms of the expansion of which in terms of δ^4 have the form [3]

$$h_a=\sqrt[3]{3q},\ u_a=h_ay-\frac{1}{2}y^2,\ v_a=\delta^2\left(\frac{1}{3}h_ay^3-\frac{2}{3}h_a^3y-\frac{1}{12}y^4\right). \tag{12}$$

Asymptotic solutions of higher order are presented in [3, 4].

Figure 1 shows the dependence of h and the mean radial velocity $u_m = q/h$ relative to the corresponding values h_α, $u_{\alpha m}$ of Eq. (12). After a formation period x_0 the solution takes on the form of Eq. (12).

During the readjustment process the velocity profiles have a complex form, an example of which is shown in Fig. 2.

The length of the formation segment x_0 and the degree to which the solution thereon differs from the asymptotic solution of Eq. (12) depends on the parameter δ the volume flow rate q(0), and the initial velocity component profiles. In studying the dependence of x_0 on δ and q(0) we considered the case $U = 3q(0)(y - y^2/2)$, $V = 0$. For the fixed value q(0) = 1/3 the length of the formation segment x_0 = 0.28, 0.90, 1.42, 2.52 at δ = 0.669, 1.000, 1.442, 3.107, respectively; for δ = 1 for initial flow rates q(0) = 0.1, 2.0, 4.0 values x_0 = 0.30, 1.44, 1.80, 2.15 were obtained. The condition $|(h - h_\alpha)/h_\alpha < 0.02$ was used as a criterion for choosing x_0.

The initial velocity component profiles have a greater effect on the form of the solution on the formation segment than on the value of x_0. For example, for the case δ = 1, q(0) = 1/3, V = 0 at $U = y - y^2/2$ the film thickness decreases monotonically with increase in x, while at $U = 2.5(y - y^2/2)^2$ the thickness curve shows local minima and maxima. The effect of the initial azimuthal velocity profile was considered for the case δ = 1, q(0) = 1, $U = \pi\sin(\pi y/2)/2$. For V = 0 the film thickness curve has a local maximum, while at V = 10U a minimum exists along with the maximum.

Thus, calculation results show that for films of relatively large thickness the radial velocity profile differs from parabolic over some initial formation segment, the length of which is comparable to the characteristic radius of the disk. The proposed method permits calculation of flows of such films.

NOTATION

Q, volume flow rate; ω, angular velocity of disk rotation; ν, kinematic viscosity of liquid; H_0, characteristic film thickness; R, minimum radius of flow region; $\delta = H_0\sqrt{\omega/\nu}$, parameter; r, θ, z, and u_r, u_θ, u_z, cylindrical coordinate system and liquid velocity components; x_0, dimensionless length of asymptotic solution formation segment.

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